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Beny Neta

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Prepared by:

BENY NETA*
Associate Professor of Mathematics

Reviewed by:

HAROLD M. FREDRICKSEN
Chairman
Department of Mathematics

Released by:

KNEALE T. MARSHALL
Dean of Information and
Policy Sciences

*Formerly associated with Texas Tech University

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There are many articles discussing the solution of boundary value problems on various parallel machines. The solution of initial value problems does not lend itself to parallelism, since in this case one uses methods that are sequential in nature.
Here we develop a parallel scheme for initial value problems cased on the box scheme and a modified recursive doubling technique.
Fully implicit Runge Kutta methods were discussed by Jackson and Norsett (1986) and Lie (1987). Lie assumes that each processor of the parallel computer having vector capabilities.

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Beny Neta

22b TELEPHONE (Include Area Code)
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SOLUTION OF LINEAR INITIAL VALUE PROBLEMS ON A HYPERCUBE

C. P. Katti
Jawaharlal Nehru University
School of Computer and Systems Sciences
New Delhi, 110067
India

and

B. Neta
Naval Postgraduate School
Department of Mathematics
Code 53Nd
Monterey, CA 94943

Abstract

There are many articles discussing the solution of boundary value problems on various parallel machines. The solution of initial value problems does not lend itself to parallelism, since in this case one uses methods that are sequential in nature.

Here we develop a parallel scheme for initial value problems based on the box scheme and a modified recursive doubling technique.

Fully implicit Runge Kutta methods were discussed by Jackson and Norsett (1986) and Lie (1987). Lie assumes that each processor of the parallel computer having vector capabilities.

1 Introduction

We consider the solution of linear initial value problems on a hypercube. "By a hypercube we intend a distributed memory MIMD computer with communication between processors ... via a communication network having the topology of a p–dimensional cube, with the vertices considered as processors and the edges as communication links" (Keller and Nelson, 1987). See also Fox (1984, 1985,

1987) and Fox and Otto (1984). Our method of solution is based on the box scheme to discretize the system of initial value problems

$$\begin{aligned} y' &= Ay + f(x) \\ y(a) &= y'_0 \end{aligned}$$

where y and f are n–dimensional vectors and A is an $n \times n$ matrix. The resulting system of equations is solved by a modified version of the recursive doubling technique (see Stone, 1973).

In the next section, the discretization is described and the resulting system of equation is given. Section 3 will describe the modified recursive doubling technique and its application to our system.

It will be interesting to experiment with the method and compare the results to a sequential initial value solver of the same order.

2 The Single Step Method

Consider the system of initial value problems

$$\begin{aligned} y' &= A(x)y + f(x), \qquad a \le x \le b \\ y(a) &= y'_0 \end{aligned} \tag{1}$$

where

$$\begin{aligned} y &= (y_1, \ldots, y_n)^T, \qquad f = (f_1(x), \ldots, f_n(x))^T, \\ y'_0 &= (y'_{10}, \ldots, y'_{n0}), \end{aligned}$$

and

$$A = a_{ij}(x), \qquad 1 \le i,\ j \le n.$$

Let

$$x_j = a + jh, \qquad j = 0, 1, \ldots, m \tag{2}$$

where

$$h = \frac{b-a}{m} \tag{3}$$

be a uniform mesh. The box scheme (see e.g. Keller, 1976), applied to (1) yields

$$\begin{aligned} y_{j+1} &= y_j + h\left\{A_{j+\frac{1}{2}}\left(y_{j+1}+y_j\right)/2 + f_{j+\frac{1}{2}}\right\}, \\ y_0 &= y_0' \end{aligned} , \tag{4}$$

where

$$\begin{aligned} A_{j+\frac{1}{2}} &= A\left(a + \left(j+\frac{1}{2}\right)h\right) \\ f_{j+\frac{1}{2}} &= f\left(a + \left(j+\frac{1}{2}\right)h\right) \end{aligned}$$

and y_j is the approximation to $y(x_j)$.

Let $\{j_i,\ i = 1,2,\ldots,s\}$ be a strictly increasing sequence such that $j_1 > 0$ and $j_s = m$. We shall compute the solution at the points $x_i = x_{j_i}$. Let Φ_i be $n \times n$ matrices defined for each i

$$\Phi_i = \prod_{j=j_{i-1}}^{j_i-1}\left(I - \frac{h}{2}\,A_{j+\frac{1}{2}}\right)^{-1}\left(I + \frac{h}{2}\,A_{j+\frac{1}{2}}\right), \qquad i = 1,2,\ldots,s, \tag{5}$$

where $j_0 = 0$ and h is sufficiently small so that $I - \frac{h}{2}\,A_{j+\frac{1}{2}}$ are nonsingular. Similarly let the n–vector φ_i be

$$\begin{aligned} \varphi_i &= \left(I - \frac{h}{2}\,A_{j_i-\frac{1}{2}}\right)^{1}\left(I + \frac{h}{2}\,A_{j_i-\frac{1}{2}}\right)\tilde{y}_{j_i-1-j_{i-1}} \\ &\quad + h\left(I - \frac{h}{2}\,A_{j_i-\frac{1}{2}}\right)^{-1} f_{j_i-\frac{1}{2}}, \qquad i = 1,2,\ldots,s, \end{aligned} \tag{6}$$

where

$$\tilde{y}_0 = 0 \tag{7}$$

and

$$\begin{aligned} \tilde{y}_{j+1} &= \left(I - \frac{h}{2}\,A_{j+\frac{1}{2}+j_{i-1}}\right)^{-1}\left[\left(I + \frac{h}{2}A_{j+\frac{1}{2}+j_{i-1}}\right)\tilde{y}_j + hf_{j+\frac{1}{2}+j_{i-1}}\right] \\ & j = 0,\ldots,j_i - j_{i-1} - 2. \end{aligned} \tag{8}$$

Then it can be easily shown as in Keller and Nelson (1987), that

$$Y_{j_i} = \Phi_i\, y_{j_{i-1}} + \varphi_i, \qquad i = 1,2,\ldots,s. \tag{9}$$

Remarks

1. The matrices to be inverted are of order n, the number of equations in the original system (1).

2. The last factor in the product defining Φ_i is the matrix required in computing φ_i.

3. The vector $\tilde{y}_{j_i-1-j_{i-1}}$ can be computed by (7) - (8) in the same loop one computes Φ_i since it requires the same matrices.

3 Parallel Evaluation

To solve (9) on a hypercube with $p = s$ processors, one can modify the recursive doubling technique developed by Stone (1973).

Let

$$\begin{aligned} b_1 &= \Phi_1 y_0' + \varphi_1 \\ b_j &= \varphi_j, \qquad j = 2,3,\ldots,s \end{aligned} \tag{10}$$

and let $Y_i(j)$ be a function of $b_j, b_{j-1}, \ldots, b_{j-i+1}, \Phi_j, \ldots, \Phi_{j-i+1}$. Then the following results can be proved using similar arguments as in Stone(1973).

Theorem. Let $Y_i(j)$ satisfy the recurrence relation

$$Y_{i+1}(j) = Y_1(j) + \Phi_j Y_i(j-1), \qquad i,j \geq 1 \tag{11}$$

with boundary conditions

$$\begin{aligned} &Y_1(j) = b_j, \quad j \geq 1 \\ &Y_i(j) = 0, \quad j \leq 0 \text{ or } i \leq 0. \end{aligned} \tag{12}$$

Then

(i)

$$Y_{i+s}(j) = Y_s(j) + \prod_{k=j-s+1}^{j} \Phi_{2j-k-s+1} Y_i(j-s) \tag{13}$$

(ii)

$$Y_i(j) = \sum_{k=1}^{j} \left\{ \prod_{s=k+1}^{j} \Phi_{j-s+k+1} \right\} Y_1(k), \qquad i \geq j \geq 1, \tag{14}$$

(iii) for

$$i \geq j \geq 1, \qquad Y_i(j) = y_{j_i}. \tag{15}$$

Corollary

$$Y_{2i}(j) = Y_i(j) + \left\{ \prod_{k=j-i+1}^{j} \Phi_{2j-k-i+1} \right\} Y_i(j-i), \quad i,j \geq 1 \tag{16}$$

This corollary provides the recursive doubling algorithm for the solution of (9). Let

$$M_i(j) = \begin{cases} \prod_{k=1}^{j} \Phi_{j-k+1} & j \leq i \\ \\ \prod_{k=j-i+1}^{j} \Phi_{2j-k+1-i}, & j \geq i \end{cases} \tag{17}$$

then (16) can be written as

$$\begin{aligned} Y_{2i}(j) &= Y_i(j) + M_i(j) Y_i(j-i) \quad && i,j \geq 1 \\ M_{2i}(j) &= M_i(j) M_i(j-i) && i,j \geq 1 \end{aligned} \tag{18}$$

with boundary conditions

$$\begin{aligned} M_1(j) &= \Phi_j, \quad j \geq 1 \\ M_1(j) &= I, \quad i \leq 0 \text{ or } j \leq 0. \end{aligned} \tag{19}$$

We are now ready to state the algorithm.

Algorithm

For $i = 1$ to $s/2$ in steps of i do:

$$\begin{aligned} Y_{2i}(j) &= Y_i(j) + M_i(j) Y_i(j-i) \quad && i \leq j \leq s \\ M_{2i}(j) &= M_i(j) M_i(j-i) && i \leq j \leq s \end{aligned}$$

Next i.

From our theorem, $Y_s(j) = y_{j_s}$ for $1 \leq j \leq s$, so that Ys is the solution of (9). We note that for each i, the indices pertaining to j are executed simultaneously on s processors. Since i doubles during each iteration, $\log_2 s$ iterations are required for computation.

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